Generalized strategies in the Minority Game

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Abstract

We show analytically how the fluctuations (i.e. standard deviation σ) in the Minority Game (MG) can decrease below the random coin-toss limit if the agents use more general, stochastic strategies. This suppression of σ results from a cancellation between the actions of a crowd, in which agents act collectively and make the same decision, and an anticrowd in which agents act collectively by making the opposite decision to the crowd.

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The Minority Game (MG) of Challet and Zhang [1–3] offers a simple paradigm for complex, adaptive systems. The MG comprises an odd number N of agents, each with s strategies and a memory size m, who repeatedly compete to be in the minority. In the basic MG, where agents always use their highest scoring strategy, the size of the fluctuations (i.e. standard deviation σ) falls below the random, coin-toss limit for large m [3]. Cavagna *et al* [4] considered an interesting modification of the basic MG, the 'Thermal Minority Game' (TMG), whereby agents choose between their strategies using an exponential probability weighting. As pointed out by Marsili *et al* [5], such a probabilistic strategy weighting has a long tradition in economics and encodes a particular behavioral model. The numerical simulations of Cavagna *et al* demonstrated that at small m, where the MG σ is larger-than-random, the TMG σ could be pushed below the random coin-toss limit just by altering this relative probability weighting, or equivalently the 'temperature' T [4]. This reduction in σ for stochastic strategies seems fairly general: for example, we had reported earlier on a modified MG in which agents with stochastic strategies also generate a smaller-than-random σ for small m [6].

In this paper, we provide a quantitative theory which explains how the standard deviation σ in the MG can get reduced from larger-than-random to smaller-than-random if more general, stochastic strategies are used. In particular, we show that stochastic strategy rules tend to *increase* the cancellation between the actions of a crowd of like-minded agents [7,8] and its anti-correlated partner (anticrowd), thereby reducing σ below the random coin-toss limit.

The MG [1] comprises an odd number of agents N who choose repeatedly between option 0 (e.g. buy) and option 1 (e.g. sell). The winners are those in the minority group, e.g. sellers win if there is an excess of buyers. The outcome at each timestep represents the winning decision, 0 or 1. A common bit-string of the m most recent outcomes is made available to the agents at each timestep [9]. The agents randomly pick s strategies at the beginning of the game, with repetitions allowed, from the pool of all possible strategies. After each turn, the agent assigns one (virtual) point to each of his strategies which would have predicted the correct outcome. In the basic MG, each agent uses the most successful strategy in his possession, i.e. the one with the most virtual points. Because of crowd effects [8], σ is larger-than-random for small m. Our task is to explain quantitatively why the larger-than-random σ in this 'crowded' regime (i.e. small m) becomes smaller-than-random when the strategy-picking rule is made stochastic, such as in Ref. [4].

Consider any two strategies r and r^* within the list of 2^{m+1} strategies in the reduced strategy space [1,8]. At any moment in the game, the strategies can be ranked according to their virtual points, $r = 1, 2 \ldots 2^{m+1}$ where $r = 1$ is the best strategy, $r = 2$ is second best, etc. In the small m regime of interest, the virtual-point strategy ranking and popularity ranking for strategies are similar [8]. Consider $s = 2$ as in Ref. [4]. Let $p(r, r^*|r^* \geq r)$ be the probability that a given agent picks r and r^*, where $r^* \geq r$ (i.e. r is the best, or equal best, among his $s = 2$ strategies). In contrast, let $p(r, r^*|r^* \leq r)$ be the probability that a given agent picks r and r^*, where $r^* \leq r$ (i.e. r is the worst, or equal worst, among his $s = 2$ strategies). Let θ be the probability that the agent uses the worst of his $s = 2$ strategies, while $1 - \theta$ is the probability that he uses the best. The probability that the agent plays r is given by

$$\begin{aligned} p_r &= \sum_{r^*=1}^{2^{m+1}} [\ \theta\ p(r, r^*|r^* \leq r) +\ (1-\theta)\ p(r, r^*|r^* \geq r)] \\ &=\ (1-\theta)\ p_+(r) +\ \theta\ p_-(r) + 2^{-2(m+1)}\ \theta \qquad (1) \end{aligned}$$

where $p_+(r)$ is the probability that the agent has picked r *and* that r is the agent's best (or equal best) strategy; $p_-(r)$ is the probability that the agent has picked r *and* that r is the agent's worst strategy. It is straightforward to show that

$$p_+(r) = \left(\left[1 - \frac{(r-1)}{2^{m+1}}\right]^2 - \left[1 - \frac{r}{2^{m+1}}\right]^2\right) \quad . \tag{2}$$

Note that $p_+(r) + p_-(r) = p(r)$ where

$$p(r) = 2^{-m}(1 - 2^{-(m+2)}) \tag{3}$$

is the probability that the agent holds strategy r after his $s = 2$ picks, with no condition on whether it is best or worst. An expression for $p_-(r)$ follows from Eqs. (2) and (3). The basic MG [1] corresponds to the case $\theta = 0$.

In the TMG, each agent is equipped at each timestep with his own (biased) coin characterised by exponential probability weightings [4]. An agent then flips this coin at each timestep to decide which strategy to use [4]. To relate the present analysis to the TMG in Ref. [4], we consider $0 \leq \theta \leq 1/2$: $\theta = 0$ corresponds to 'temperature' $T = 0$ while $\theta \to 1/2$ corresponds to $T \to \infty$ [10] with $\theta = 1/2[1 - \tanh(1/T)]$. Consider the mean number of agents playing strategy r which is given by

$$n_r = Np_r = N\ (1-2\theta)\ p_+(r) + N\ \theta\ p(r) + 2^{-2(m+1)}\ N\ \theta \quad . \tag{4}$$

If n_r agents use the same strategy r, then they will act as a 'crowd', i.e. they will make the same decision. If $n_{\bar{r}}$ agents simultaneously use the strategy $\bar{r}$ anticorrelated to r, they will make the opposite (anticorrelated) decision and will hence act as an 'anticrowd' [8]. Averaging over the stochastics of the strategy-use which is introduced by allowing each agent to flip a (biased) coin at each timestep, the standard deviation σ_θ in the number of agents making a particular decision (say 0) becomes

$$\sigma = \left[\frac{1}{2}\sum_{r=1}^{2^{m+1}}\sum_{r'=1}^{2^{m+1}}\frac{1}{4}|n_r - n_{r'}|^2 P(r' = \bar{r})\right]^{\frac{1}{2}} \tag{5}$$

where $P(r' = \bar{r})$ is the probability that any strategy r' is the anti-correlated partner of strategy r in the list of strategies when ordered in terms of popularity $\{n_r\}$ [8]. The quantities n_r and $n_{r'}$ are θ-dependent (see Eq. (4)). Substituting Eqs. (3) and (4) for r and r' into Eq. (5) yields

$$\sigma_\theta = |1 - 2\theta|\ \sigma_{\theta=0} \tag{6}$$

where $\sigma_{\theta=0}$ is the standard deviation when $\theta = 0$, i.e. the basic MG. Equation (6) explicitly shows that the standard deviation σ_θ *decreases* as θ increases (recall $0 \leq \theta \leq 1/2$): in other words, the standard deviation decreases as agents use their worst strategy with increasing probability. An increase in θ leads to a reduction in the size of the larger crowds using high-scoring strategies, as well as an increase in the size of the smaller anticrowds using lower-scoring strategies, hence resulting in a more substantial cancellation effect between the crowd and the anticrowd. As θ increases, σ_θ will eventually drop *below* the random coin-toss result at $\theta = \theta_c$ where

$$\theta_c = \frac{1}{2} - \frac{\sqrt{N}}{4}\frac{1}{\sigma_{\theta=0}} \quad . \tag{7}$$

Elsewhere we presented a quantitative formulation for $\sigma_{\theta=0}$. In particular, Ref. [8] showed that $P(r' = \bar{r})$ at small m lay between that of a δ-function distribution $\delta_{r',2^{m+1}+1-r}$ peaked at $r' = 2^{m+1} + 1 - r$, and a flat distribution $[2^{(m+1)} - 1]^{-1}$ (N.B. $P(r' = \bar{r}) = 0$ at $r' = r$) [8]. Using these distributions, the resulting expressions for σ_θ are

$$\sigma_{\rm delta} = (1-2\theta)\frac{N}{\sqrt{3}}2^{-(\frac{m}{2}+1)}\left[1-2^{-2(m+1)}\right]^{\frac{1}{2}} \quad (8)$$

and

$$\sigma_{\rm flat} = (1-2\theta)\frac{N}{2\sqrt{3}}[2^{(m+1)}-1]^{-\frac{1}{2}}\left[1-2^{-2(m+1)}\right]^{\frac{1}{2}} \quad (9)$$

respectively. For the TMG, $\theta = (1/2)[1-\tanh(1/T)]$ although we emphasize that the present theory is not limited to the case of 'thermal' strategy weightings.

Figure 1 shows a comparison between the theory of Eqs. (8) and (9) and numerical simulation for various runs. The theory agrees well in the range $\theta = 0 \to 0.35$ and, most importantly, provides a quantitative explanation for the transition in σ from larger-than-random to smaller-than-random as θ (and hence T) is increased. The numerical data for different runs has a significant natural spread: remarkably most of these data points lie in the region of these two analytic curves.

Above $\theta = 0.35$, the numerical data tend to flatten off while the present theory predicts a decrease in σ as $\theta \to 0.5$. This is because the present theory averages out the fluctuations in strategy-use at each time-step (Eq. (4) only considers the mean number of agents using strategy r). Consider $\theta = 0.5$. For a particular configuration of strategies picked at the start of the game, and at a particular moment in time, the number of agents using each strategy is typically distributed *around* the mean value $n_r = N2^{-(m+1)}$ given by Eq. (4) for $\theta = 0.5$. The resulting distribution describing the strategy-use is therefore non-flat. It is these fluctuations about the mean n_r and $n_{\bar{r}}$ which give rise to a non-zero σ. Our present crowd-anticrowd theory can be extended to account for the effect of these fluctuations in strategy-use for $\theta \to 0.5$ in the following way: All N agents are randomly assigned 2 strategies. To represent a turn in the game, each agent flips a (fair) coin to decide which of the two strategies is the preferred one. Having generated a list of the number of agents using each strategy, σ is then found in the usual way by cancelling off crowds and anticrowds. A time-averaged value for σ is then obtained by averaging over 100 independent coin-flip outcomes for the given initial distribution of strategies among agents. This procedure provides a semi-analytic calculation for the value of σ at $\theta = 0.5$. Inset (a) in Fig. 1 shows the measured numerical distribution in σ for $\theta = 0.5$, while inset (b) shows the result from the semi-analytic procedure. The two distributions are in good agreement. It is also possible to perform a fully analytic calculation of the average σ_θ in the $\theta \to 0.5$ limit: the initial random assignment of strategies can be modelled using a random-walk. This yields an average value of $|n_r - n_{\bar{r}}|^2$ (i.e. time-averaged over the fluctuations) given by $N2^{-m}(1-2^{-(m+1)})$. Summing over all pairs of correlated-anticorrelated strategies, i.e. all crowd-anticrowd pairs, yields a (configuration-average) standard deviation given by

$$\sigma_{\theta\to 0.5} = \frac{\sqrt{N}}{2}[1-2^{-(m+1)}]^{\frac{1}{2}} \quad . \quad (10)$$

For $N = 101$ as shown in Fig. 1, the analytic theory (Eq. (10)) and semi-analytic theory both give an average standard deviation $\sigma_{\theta\to 0.5} = 4.7$, while the numerical simulation yields 4.5. The agreement is good. We note that all these values lie *below* the random coin-toss limit $\sqrt{N}/2 = 5.0$: hence the theoretical and numerical results are *not* equal to the random coin-toss limit as $\theta \to 0.5$ (i.e. $T \to \infty$). We have also checked this for $N = 11$: here the analytic value is 1.55, the semi-analytic value is 1.56 and the numerical value is 1.55, however the random coin-toss limit is 1.66. We note that the numerical results obtained using the full and reduced strategy spaces are very similar; hence our results and conclusions are indeed general. Finally, we would like to point out that the high-T regime has been the subject of much debate recently. In particular, Challet *et al* identified problems [11–13] with the $T \to \infty$ results of Ref. [4]. We refer to Refs. [11–13] for a detailed discussion.

Given the complexity of these many-agent games, it is remarkable that our simple analytic approach invoking crowd effects can quantitatively explain the main feature whereby σ falls below the random, coin-toss limit as θ (or T [4]) increases. This feature, which is arguably the most striking result of the TMG, can therefore be explained without having to solve the full game dynamics. This result also strengthens our belief that many properties of the MG can be understood using simple notions of crowd-anticrowd interplay.

REFERENCES

[1] D. Challet and Y.C. Zhang, Physica A **246**, 407 (1997); *ibid.* **256**, 514 (1998); *ibid.* **269**, 30 (1999).

[2] D. Challet and M. Marsili, Phys. Rev. E **60**, R6271 (1999); D. Challet, M. Marsili, and R. Zecchina, Phys. Rev. Lett. **84**, 1824 (2000); D. Challet and M Marsili, cond-mat/9908480.

[3] R. Savit, R. Manuca and R. Riolo, Phys. Rev. Lett. **82**, 2203 (1999).

[4] A. Cavagna, J.P. Garrahan, I. Giardina and D. Sherrington, Phys. Rev. Lett. **83**, 4429 (1999); see also J.P. Garrahan, E. Moro and D. Sherrington, cond-mat/0004277.

[5] M. Marsili, D. Challet and R. Zecchina, cond-mat/9908480. T-dependent, Boltzmann-like strategy weightings were discussed by M. Marsili at the International Workshop on Econophysics and Statistical Finance (Palermo, September 1998); see Physica A **269**, 9 (1999).

[6] N.F. Johnson, P.M. Hui, R. Jonson and T.S. Lo, Phys. Rev. Lett. **82**, 3360 (1999).

[7] N.F. Johnson, M. Hart and P.M. Hui, Physica A **269**, 1 (1999).

[8] M. Hart, P. Jefferies, N.F. Johnson and P.M. Hui, cond-mat/0003486.

[9] See D. Challet and M. Marsili, cond-mat/0004196 and references therein for demonstrations confirming the relevance of the actual memory in the MG, in contrast to the claim of A. Cavagna [Phys. Rev. E **59**, R3783 (1999)].

[10] The Thermal Minority Game discussed in Ref. [4] depends on a parameter T (or equivalently $1/\beta$) called a 'temperature'. We could similarly define T by setting the probability of playing the worst strategy $\theta = e^{-\beta}/(e^{\beta} + e^{-\beta})$. Hence $T = 2[\ln(\theta^{-1} - 1)]^{-1}$. $T = 0$ corresponds to $\theta = 0$ while $T \to \infty$ corresponds to $\theta \to 1/2$, hence we will only consider $0 \leq \theta \leq 1/2$ in this paper.

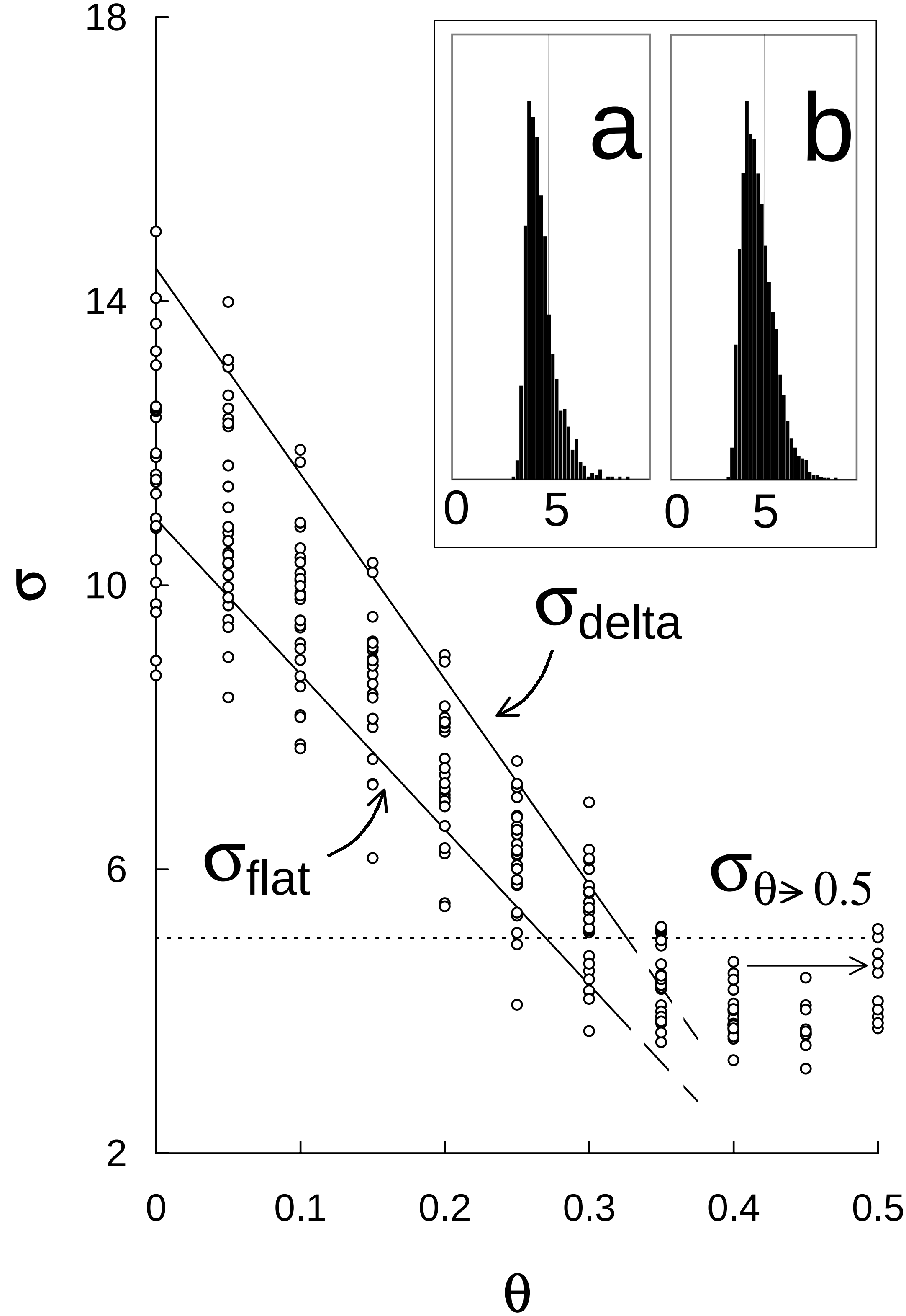
[11] D. Challet, M. Marsili and R. Zecchina, cond-mat/0004308.

[12] A. Cavagna, J.P. Garrahan, I. Giardina and D. Sherrington, cond-mat/0005134.

[13] M. Marsili and D. Challet, Adv. Complex Systems **1**, 1 (2000).

Figure Captions

Figure 1: Comparison between numerical simulations (circles) and the present analytic theory for the standard deviation σ as a function of the probability θ. Analytic theory (solid line): $\sigma_{\rm delta}$ using Eq. (8), $\sigma_{\rm flat}$ using Eq. (9). $N = 101$, $m = 2$ and $s = 2$. Dashed line shows random coin-toss value. Solid arrow indicates theoretical value $\sigma_{\theta \rightarrow 0.5} = 4.7$ for $\theta \rightarrow 0.5$ (see text). Inset shows distribution of σ values at $\theta = 0.5$ for several thousand randomly-chosen initial strategy configurations: (a) numerical simulation, (b) semi-analytic theory (see text). Quantities shown are dimensionless.



18
14
10
6
2
σ
0
0.1
0.2
0.3
0.4
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θ
σ_{delta}
σ_{flat}
$\sigma_{\theta > 0.5}$
a
b
0
5
0
5